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Akin $N = 2$ SUSY Yang Mills Theories and Instanton Expansion

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Abstract

The low energy effective actions of the $N = 2$ SUSY $SU(N_c)$ QCD are considered at the symmetric point on the moduli space. The classes of such theories have similar spectral curves. This fact allows us to show that all these models have the same structure of the coupling matrix and to show that the $N_f = 2N_c$ spectral curve can not be presented as a double covering of the sphere. We calculate first instanton contributions to the coupling matrix and get nonperturbative β-functions in the $SU(2)$ gauge theory with non-zero bare masses of the matter hypermultiplets.

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1 Introduction

Many exact results have been obtained since the discovery by Seiberg and Witten [1, 2] of the exact solution of the low energy $N = 2$ supersymmetric $SU(2)$ gauge theory in four dimensions. The low energy $N = 2$ SUSY Yang-Mills theories contain $g = N_c - 1$ abelian $N = 2$ vector supermultiplets, which can be decomposed into g $N = 1$ chiral multiplets A_i plus g $N = 1$ vector multiplets W_i. According to [1, 2] the scalar component a_i of the A_i (namely, its vacuum expectation value (vev)) is the local coordinate on the quantum moduli space of the effective action of the theory. It is given by the integrals of meromorphic differentials λ over the basic cycles α_i and β_i, such that $\alpha_i \circ \beta_j = \delta_{ij}$, on a hyper-elliptic curve $\mathcal{C}$. In particular, their derivatives with respect to the symmetric vevs $s_k = (-1)^k \sum_{i_1 < \cdots < i_k} a_{i_1} \cdots a_{i_k}$ $(k = 2, \cdots, N_c)$ are equal to [3]:

$$\begin{aligned}\frac{\partial a_i}{\partial s_k} &\sim A_{ki} = \int_{\alpha_i} \frac{x^{N_c-k}dx}{y} \\ \frac{\partial a_D i}{\partial s_k} &\sim B_{ki} = \int_{\beta_i} \frac{x^{N_c-k}dx}{y}\end{aligned} \tag{1}$$

We will be mainly interested in the coupling matrix T_{ij}, which is the period matrix on $\mathcal{C}$. In the matrix form, it can be presented as [3]:

$$\mathbf{T} = \mathbf{A}^{-1}\mathbf{B} \tag{2}$$

In some particular cases the integration in (1) may be performed explicitly. It has been done for the $SU(2)$ [3, 4, 5] and $SU(3)$ groups [3]. For higher N_c, it is not so immediate, and, therefore, there were a series of calculations taking into account only the one - instanton corrections [6]. For $N_c > 3$, integration in (1) becomes much more complicated, and Picard - Fuchs equations are not known for arbitrary N_c and N_f.

Some theories have the same integrals - this allows us to derive relations between couplings (see Section 2) and, in some particular cases, even to obtain exact results. The first step in this direction was done in [7], where some exact beta functions for the $SU(2)$ and $SU(3)$ cases were calculated.

In Section 2, we discuss the coupling constants for the theories with N_c colors and N_f ($N_f = 0$ or $N_f = N_c$) massless flavours. We calculate them for the second order in the instanton expansion by the direct evaluation of integrals (1).

All the bare masses m_k of the matter hypermultiplets are put zero (till the Appendix).

As later we deal with instanton contributions to the prepotential, it is relevant to remind the perturbative expansion of the prepotential which is saturated in one loop due to the supersymmetry:

$$\mathcal{F} = i\frac{2N_c - N_f}{8\pi} \sum_{i<j} (A_i - A_j)^2 \log \frac{(A_i - A_j)^2}{\Lambda^2} \tag{3}$$

In Section 3 we consider the scale invariant $N_f = 2N_c$ theory. It has the classical period matrix $\mathbf{T}$ proportional to the matrix $\mathbf{C}$: $C_{ij} = \delta_{ij} + 1$ (in the basis $A_i = a_i$, $i = 1 \cdots g$; $A_{N_c} = -\sum_{i=1}^{g} a_i$). By comparing the spectral curves for this theory and for the $N_f = N_c$ one, we demonstrate that spectral curve for UV finite $N_f = 2N_c$ ($N_c > 2$) theory can not be hyperelliptic (double covering of CP^1).

Section 4 is devoted to the $SU(2)$ theory. We present the strong evidence for the method proposed by J. A. Minahan and D. Nemeschansky which helps one to obtain some useful relations between the $N_f = 0$ and $N_f = 2$ theories.

In Appendix we present some nonperturbative β-functions of the $SU(2)$ gauge theory with non-zero masses of the matter hypermultiplets.

2 General N_c

Now we are going to discuss some akin $N = 2$ SUSY Yang-Mills theories which have the similar spectral curves each depending on one dimensionless parameter. Namely, we note that the curves for the $N_f = 0$ and $N_f = N_c$ theories have the same forms in the symmetric point on the moduli space (compare (4) and (5)). The period matrix T_{ij} must be the same for the both theories, turning into each other by an appropriate replace of the parameters. Before going further, let us stress that the curves are regarded akin if they are related by $SL(2, \mathbf{C})$ transformation (this is the common property of the two-dimensional manifolds) or by rescaling x and y (because we restrict ourselves to the only parameter and T_{ij} is dimensionless too).

For instance, for general $N_c > 2$ with all the order parameters s_k being zero but $s_{N_c} = -u \neq 0$, the $N_f = 0$ and $N_f = N_c$ curves take the forms [5]:

- $N_f = 0$

$$y^2 = \left(x^{N_c} - u^{(0)}\right)^2 - \Lambda^{(0)2N_c} \Leftrightarrow y^2 = x^{2N_c} - 2F^{(0)}x^{N_c} + 1 \tag{4}$$

- $N_f = N_c$

$$y^2 = \left(x^{N_c} - u^{(N_c)} + \frac{\Lambda^{(N_c)N_c}}{4}\right)^2 - \Lambda^{(N_c)N_c}x^{N_c} \Leftrightarrow y^2 = x^{2N_c} - 2F^{(N_c)}x^{N_c} + 1 \tag{5}$$

where

- $N_f = 0$

$$F^{(0)} = \frac{u^{(0)}}{\sqrt{u^{(0)2} - \Lambda^{(0)2N_c}}} \Leftrightarrow \Lambda^{(0)2N_c} = u^{(0)2}(1 - F^{(0)-2}) \tag{6}$$

- $N_f = N_c$

$$F^{(N_c)} = \frac{u + \frac{\Lambda^{(N_c)N_c}}{4}}{u - \frac{\Lambda^{(N_c)N_c}}{4}} \Leftrightarrow \Lambda^{(N_c)N_c} = 4u^{(N_c)}\frac{F^{(N_c)} - 1}{F^{(N_c)} + 1} \tag{7}$$

As a consequence, their couplings are (here we denote $v = \dfrac{\Lambda^{N_c}}{u}$):

$$\begin{aligned} T_{ij}^{(N_c)}(v^{(N_c)}) &= T_{ij}^{(0)}\left(\sqrt{\frac{v^{(N_c)}}{(1 + \frac{v^{(N_c)}}{4})^2}}\right); \\ T_{ij}^{(0)}(v^{(0)}) &= T_{ij}^{(N_c)}\left(\frac{8}{v^{(0)2}}\left(1 - \frac{v^{(0)2}}{2} - \sqrt{1 - v^{(0)2}}\right)\right), N_c > 2 \end{aligned} \tag{8}$$

The same procedure may be performed for other gauge groups. One does not need to know integrals (1) explicitly. Instead, it is sufficient to compare the curves, i. e. to obtain relations between the quantities which may be $\mathbf{T}$, β, $\mathcal{F}$ or their expansions in series at small v.

Let us evaluate (1) to find period matrix, namely, first instanton terms. We will focus on the $N_f = 0$ theory (as it has been shown earlier $N_f = N_c$ case is completly analogous):

$$A_{kl} = 2\int_{x_{-,l}}^{x_{+,l}} \frac{x^{N_c-k}dx}{y} = -\frac{2\pi i}{N_c}\epsilon^{l(1-k)}z_-^{-m}F_{2,1}(\frac{1}{2}, m; 1; 1 - \frac{z_+}{z_-}) \tag{9}$$

$$B_{kl} = 2\int_{x_{-,1}}^{x_{-,l}} \frac{x^{N_c-k}dx}{y} = \frac{2}{N_c}(\epsilon^{l(1-k)} - \epsilon^{(1-k)})\frac{z_-^{\frac{1}{2}-m}}{z_+^{\frac{1}{2}}}\frac{\Gamma(1-m)\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2}-m)}F_{2,1}(\frac{1}{2}, 1-m; \frac{3}{2}-m; \frac{z_-}{z_+}) \tag{10}$$

where we chose the cycles as shown on Fig.1 ($N_c = 4$ assumed) so that:

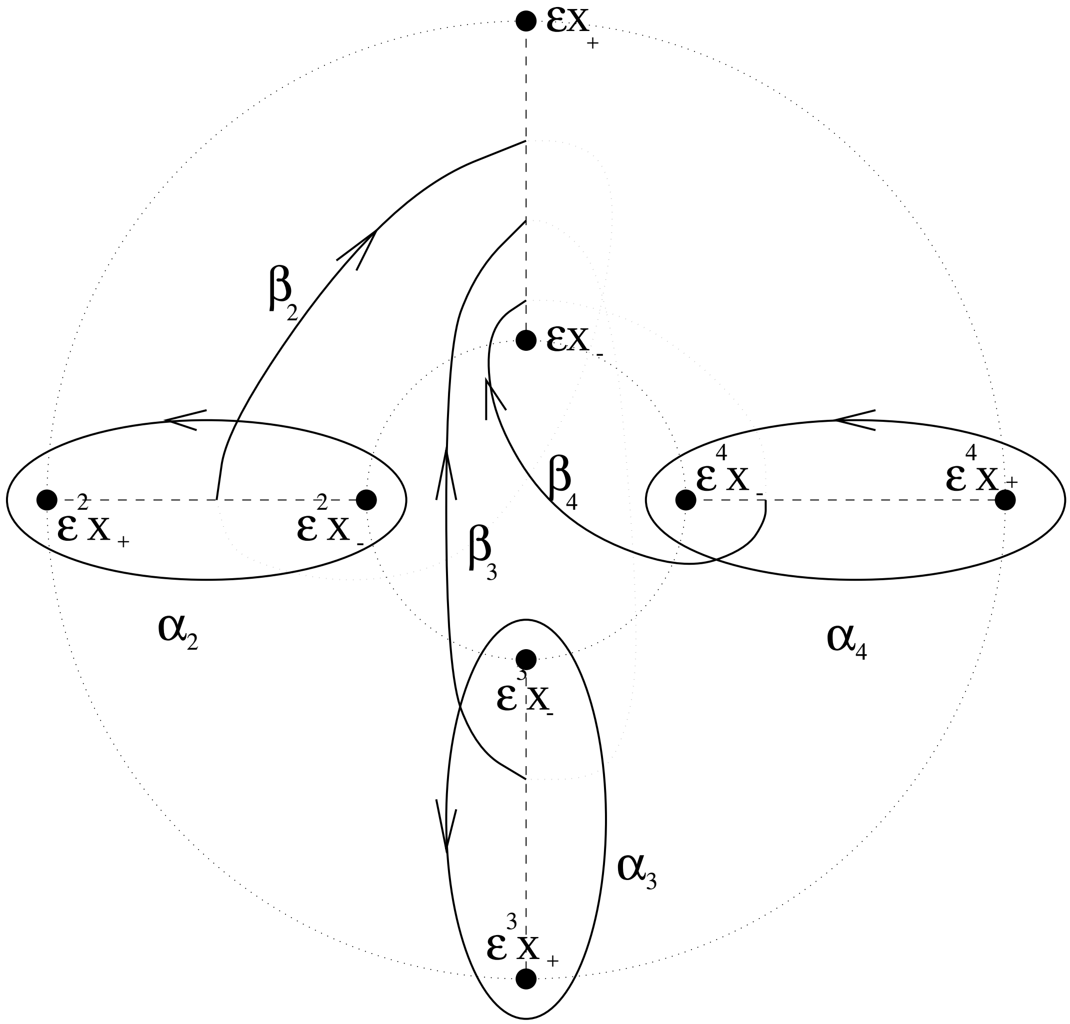

Figure 1: The homology cycles for $N_c = 4$

$$x_{\pm,l} = \epsilon^l x_\pm ; \epsilon = \exp\left(\frac{2\pi i}{N_c}\right); m = \frac{k-1}{N_c} \tag{11}$$

$$x_\pm = (u \pm \Lambda^{N_c})^{\frac{1}{N_c}} = z_\pm^{\frac{1}{N_c}}; \tag{12}$$

$$\lambda = \frac{z_+}{z_-} - 1; \tag{13}$$

In the weak coupling limit ($|\lambda| \ll 1$), (9) and (10) take the forms:

$$A_{kl} = -\frac{2\pi i}{N_c}\epsilon^{l(1-k)} z_-^{-m}(1 - \frac{m}{2}\lambda + \frac{3}{16}m(m+1)\lambda^2 + (\frac{1}{8} - \frac{3}{32}m - \frac{1}{6}m^2)m\lambda^3 + \cdots) \tag{14}$$

$$B_{kl} = \frac{2}{N_c}(\epsilon^{l(1-k)} - \epsilon^{(1-k)}) z_-^{-m}((\ln\frac{4}{\lambda} - C - \psi(1-m))(1 - \frac{m}{2}\lambda + \frac{3}{16}m(m+1)\lambda^2 -$$

$$- \frac{5}{96}m(m+1)(m+2)\lambda^3) + \frac{\lambda}{2} + \frac{m^2-5m-3}{16}\lambda^2 + (\frac{5}{48} + \frac{1}{32}m(8+2m-m^2))\lambda^3 + \cdots) \tag{15}$$

where C is the Euler constant, and $\psi(x) = \frac{\Gamma'}{\Gamma}$ is the logarithmic derivative of the gamma function. It is useful to note that the dependence of $\mathbf{A}$ and $\mathbf{B}$ on k and l is :

$$A_{kl} = a(k)E_{kl} \qquad E_{kl} = \epsilon^{l(1-k)} \tag{16}$$

$$B_{kl} = b(k)(\epsilon^{l(1-k)} - \epsilon^{(1-k)}) = b(k)(E_{kl} + \sum_l E_{kl}) \tag{17}$$

Substituting it into (2), one gets the matrix of coupling constants for the $N_f = 0$ theory:

$$T_{ij} = \sum_k \frac{b(k)}{a(k)}(E^{-1})_{ik}(E_{kj} + \sum_j E_{kj}) = \tag{18}$$

$$= \frac{i}{\pi}\sum_k (E^{-1})_{ik}(E_{kj} + \sum_j E_{kj})(\ln\frac{4}{\lambda} - C - \psi(1-m) + \frac{\lambda}{2} - \frac{3}{16}\lambda^2 + \cdots)$$

It is easy to show that $\mathbf{T}$ is proportional to $\mathbf{C}$, if and only if all the exact $a(k)^{-1}b(k)$ are the same functions for any k. Imposing the constraint $T_{ii} = 2T_{i\neq j}$ on $\mathbf{T}$, one immediately gets from (18):

$$\sum_k \frac{b(k)}{a(k)}(E^{-1})_{ik}E_{kj}|_{i\neq j} = 0$$

which, after simple transformations, turns into :

$$a^{-1}(k)b(k) = const \tag{19}$$

This equation must be satisfied in all orders in λ. Obviously, it is not true and $\mathbf{T} \not\sim \mathbf{C}$ in symmetric point.

3 $N_f = 2N_c$ case

When $N_f = 2N_c$ and the bare masses are zero we get conformally invariant theory. It has the classical period matrix $\mathbf{T}$ proportional to the matrix $\mathbf{C}$: $T_{ij} = \tau C_{ij} = \tau(\delta_{ij} + 1)$.

The spectral curve for this case was proposed in [5] and, when the masses are set to zero, curve reads ($s_i = 0$, $i \neq N_c$; $s_{N_c} = -u \neq 0$): [1]

$$y^2 = \left[(1 + \frac{L}{4})x^{N_c} - ul\right]^2 - Lx^{2N_c} \Leftrightarrow y^2 = x^{2N_c} - 2Fx^{N_c} + 1 \tag{20}$$

[1] L and l are modular forms expressed through the higher genus θ-constants defined on $\tau\mathbf{C}$

This curve has the same form as (4) and (5), with the associated period matrix $\mathbf{T} \sim \mathbf{C}$ having the same structure. From the computation of Section 2 one can easily see that this is not the case. Moreover, it may be seen from (3) that even the perturbative coupling matrix is not proportional to the matrix $\mathbf{C}$ anywhere on the moduli space if $N_c > 2$ and $N_f < 2N_c$ (see also [7]).

The simplest way to see it is to compute $\mathbf{T}$ in basis $a_i = A_i - A_{N_c}$, $i = 1 \cdots g$. Requirement for T_{ij} to be proportional to the classical matrix leads to the constraint on a_i :

$$(g-1)\log a_i = \sum_{k\neq i}^{g} \log(a_i - a_k)$$

which must be satisfied for any i. Since these equations have not nontrivial solutions, we come to the statement that $\mathbf{T} \not\sim \mathbf{C}$ at any point on the moduli space.

Furthemore, let us suppose that the $N_f = 2N_c$ curve is written as a polynomial of power $2N_c$ (for instance, as it was proposed in [5]). One can compare the spectral curve for such a theory and for the $N_f = N_c$ one. Since the both theories have the spectral curves which are polynomials of power $2N_c$ and there are $3N_c - 2$ parameters ($s_k^{(N_c)}$, $s_k^{(2N_c)}$ and $m_k^{(N_c)}$), one can adjust them so that the curves are getting identical (up to $SL(2,C)$ transformations), and the corresponding theories have the same structure of the coupling matrices. But from the previous arguments based on perturbative results, we know that it does not take place. Hence, the $N_f = 2N_c$ spectral curve is not a polynomial of power $2N_c$.

Thus, we demonstrate that, for $N_f = 2N_c$, the spectral curve can not be hyperelliptic surface (the double covering of CP^1). Our conjecture is that, for scale invariant theories, covering of the sphere must be replaced by covering of elliptic curve with natural elliptic parameter $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.

4 More on $N_c = 2$

Let us repeat the same procedure for the $SU(2)$ group in detail. Analogously:

$$T^{(N_c)}(v^{(N_c)}) = T^{(0)}\left(\sqrt{\frac{v^{(N_c)}(1+\frac{v^{(N_c)}}{8})}{(1+\frac{3v^{(N_c)}}{8})^2}}\right);$$

$$T^{(0)}(v^{(0)}) = T^{(N_c)}\left(8\frac{1-\sqrt{1-v^{(0)2}}}{1+3\sqrt{1-v^{(0)2}}}\right), N_c = 2 \tag{21}$$

We use it to relate the coefficients $\mathcal{F}_k$ of the instanton expansion for the prepotential:

$$\mathcal{F}(a) = \frac{ia^2}{4\pi}\left[b\ln\left(\frac{a}{\Lambda}\right) + \sum_{k=1}^{\infty}\mathcal{F}_k(N_f)\left(\frac{\Lambda}{a}\right)^{kb}\right] \tag{22}$$

where $b = 4 - N_f$. The order parameter is known from the Picard-Fuchs equation [8]:

$$u = \frac{4\pi}{ib}\left(a\frac{\partial\mathcal{F}}{\partial a} - 2\mathcal{F}\right) = a^2\left[1 - \sum_{k=1}^{\infty}k\mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{kb}\right] \tag{23}$$

T appears as the second derivative of the prepotential :

$$T = \frac{i}{2\pi}\left[b\left(\frac{3}{2} + \ln\left(\frac{a}{\Lambda}\right)\right) + \sum_{k=1}^{\infty}(1-\frac{kb}{2})(1-kb)\mathcal{F}_k\left(\frac{\Lambda}{a}\right)^{kb}\right] \tag{24}$$

Substituting $N_f = 0$ and $N_f = 2 (= N_c)$ and inverting the series for u, one gets a:

- $N_f = 0$

$$\frac{a^2}{\Lambda^2} = \frac{u}{\Lambda^2}\left[1 + \mathcal{F}_1\left(\frac{\Lambda^2}{u}\right)^2 + (2\mathcal{F}_2 - \mathcal{F}_1^2)\left(\frac{\Lambda^2}{u}\right)^4 + (3\mathcal{F}_3 - 8\mathcal{F}_1\mathcal{F}_2 + 2\mathcal{F}_1^3)\left(\frac{\Lambda^2}{u}\right)^6 + \cdots\right] \tag{25}$$

- $N_f = 2$

$$\frac{a^2}{\Lambda^2} = \frac{u}{\Lambda^2}\left[1 + \mathcal{F}_1\left(\frac{\Lambda^2}{u}\right) + 2\mathcal{F}_2\left(\frac{\Lambda^2}{u}\right)^2 + (3\mathcal{F}_3 - 2\mathcal{F}_1\mathcal{F}_2)\left(\frac{\Lambda^2}{u}\right)^3 + \cdots\right] \tag{26}$$

After insertion of these results into (24), we obtain the instanton expansion for T ($v = \frac{\Lambda^2}{u}$):

- $N_f = 0$

$$T^{(0)} = \frac{i}{2\pi}\left[6 + \ln\frac{a^2}{\Lambda^2} + 3\mathcal{F}_1^{(0)}\left(\frac{\Lambda^2}{a^2}\right)^2 + 21\mathcal{F}_2^{(0)}\left(\frac{\Lambda^2}{a^2}\right)^4 + 55\mathcal{F}_3^{(0)}\left(\frac{\Lambda^2}{a^2}\right)^6 + \cdots\right] =$$
$$= \frac{i}{2\pi}\left[6 - \ln v^{(0)2} + 4\mathcal{F}_1^{(0)}v^{(0)2} + (23\mathcal{F}_2^{(0)} - \frac{15}{2}\mathcal{F}_1^{(0)2})v^{(0)4} + \cdots\right] \tag{27}$$

- $N_f = 2$

$$T^{(2)} = \frac{i}{2\pi}\left[3 + \ln\frac{a^2}{\Lambda^2} + 3\mathcal{F}_2^{(2)}\left(\frac{\Lambda^2}{a^2}\right)^2 + 10\mathcal{F}_3^{(2)}\left(\frac{\Lambda^2}{a^2}\right)^3 + \cdots\right] =$$
$$= \frac{i}{2\pi}\left[3 - \ln v^{(2)} + \mathcal{F}_1^{(2)}v^{(2)} + (5\mathcal{F}_2^{(2)} + \frac{1}{2}\mathcal{F}_1^{(2)2})v^{(2)2} + \cdots\right] \tag{28}$$

In order to check (21), one must substitute

$$8\frac{1 - \sqrt{1 - v^{(0)2}}}{1 + 3\sqrt{1 - v^{(0)2}}} = v^{(0)2}\left(1 + \frac{5}{8}v^{(0)2} + \frac{29}{64}v^{(0)4} + \cdots\right)$$

into (28):

$$T^{(0)}(v^{(0)}) = T^{(N_c)}\left(8\frac{1 - \sqrt{1 - v^{(0)2}}}{1 + 3\sqrt{1 - v^{(0)2}}}\right) =$$
$$\frac{i}{2\pi}\left[3 - \ln v^{(0)2} + (\mathcal{F}_1^{(2)} - \frac{5}{8})v^{(0)2} + (5\mathcal{F}_2^{(2)} + \frac{5}{8}\mathcal{F}_1^{(2)} + \frac{1}{2}\mathcal{F}_1^{(2)2} - \frac{1}{16})v^{(0)4} + \cdots\right] \tag{29}$$

Comparison with the coefficients in (27) yields the system of equations for the first two of them:

$$\left\{\begin{array}{c} 4\mathcal{F}_1^{(0)} = \mathcal{F}_1^{(2)} - \frac{5}{8} \\ 23\mathcal{F}_2^{(0)} - \frac{15}{2}\mathcal{F}_1^{(0)2} = 5\mathcal{F}_2^{(2)} + \frac{5}{8}\mathcal{F}_1^{(2)} + \frac{1}{2}\mathcal{F}_1^{(2)2} - \frac{1}{16} \end{array}\right.$$

It may seem not to be so interesting to deal with these equations, since one can get explicit expressions for the coefficients in terms of θ-constants for these theories, nevertheless it allows to express immediately $N_f = 2$ instanton terms through $N_f = 0$ ones and provides the good consistency check for them. Let us mention that the results of [8] do not satisfy them.

Now let us prove the exact formula expressing β through θ - constants [2] [7]:

$$\beta^{(0)} = \frac{2}{\pi i}\frac{\theta_3^4(2T) + \theta_2^4(2T)}{\theta_4^8(2T)}$$
$$\beta^{(2)} = \frac{1}{2\pi i}\frac{\theta_3^4(2T) + \theta_4^4(2T)}{\theta_3^4(2T)\theta_4^4(2T)} \tag{30}$$

The $N_f = 4$ curve is [5]:

$$y^2 = x^4 - 2Fx^2 + 1$$
$$F = \frac{\theta_2^4(T) + \theta_3^4(T)}{\theta_2^4(T) - \theta_3^4(T)} \tag{31}$$

[2] $\theta_2 = \theta[\frac{1}{2}, 0] = \sum_{n\in Z} q^{(n+\frac{1}{2})^2}$, $\theta_3 = \theta[0,0] = \sum_{n\in Z} q^{n^2}$, $\theta_4 = \theta[0, \frac{1}{2}] = \sum_{n\in Z}(-1)^n q^{n^2}$

where x and y are rescaled by the vev u. Of course, a, a_D, A and B depend on u but T. So we know the connection between the factor F in equation (31) for u and T. Let us show it by direct calculation for the $N_f = 0$ curve [5]:

$$y^2 = (x^2 - u)^2 - \Lambda^4 \tag{32}$$

This equation may be rewritten in the form :

$$y^2 = x^4 - 2\frac{u}{\sqrt{u^2 - \Lambda^4}}x^2 + 1 \tag{33}$$

from which (using (31)) we get $\Lambda^4 = u^2(1 - F^{-2})$ (the result of [7]), or, finally:

$$\Lambda^4 = u^2(1 - F^{-2}) = u^2 \left(\frac{2\theta_2^2\theta_3^2}{\theta_2^4 + \theta_3^4}\right)^2 \tag{34}$$

One can easily integrate (1) in terms of hyper-geometric functions:

$$A = 2\int_{x_+}^{x_-} \frac{dx}{\sqrt{(x^2-u)^2 - \Lambda^4}} = -\frac{2\pi i}{x_+ + x_-}F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; \frac{(x_+ - x_-)^2}{(x_+ + x_-)^2})$$
$$B = 2\int_{-x_+}^{x_+} \frac{dx}{\sqrt{(x^2-u)^2 - \Lambda^4}} = \frac{2\pi}{x_-}F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{x_+}{x_-}\right)^2) \tag{35}$$

where $x_+ = \sqrt{u + \Lambda^2}$ and $x_- = \sqrt{u - \Lambda^2}$ are roots of (32). Note also that B has the form:

$$B = \frac{2\pi}{x_-}F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{x_+}{x_-}\right)^2) = \frac{2\pi}{x_+ + x_-}F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; \frac{4x_+x_-}{(x_+ + x_-)^2}) \tag{36}$$

As it was mentioned above, T is the ratio of periods:

$$T = i\frac{F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; 1 - w)}{F_{2,1}(\frac{1}{2}, \frac{1}{2}; 1; w)}$$

with $w = \frac{(x_+ - x_-)^2}{(x_+ + x_-)^2}$. Solution to this equation is known (see [9] for example):

$$w = \frac{\theta_2^4(0, q)}{\theta_3^4(0, q)} \Rightarrow u - \Lambda^2 = (u + \Lambda^2)\left(\frac{\theta_3^2 - \theta_2^2}{\theta_3^2 + \theta_2^2}\right)^2$$

which leads to the result (34) after simple transformations. We have obtained it by comparing this $N_f = 0$ curve (32) and that with $N_f = 4$ (31) (the method by Minahan and Nemechansky [7]) and by direct calculation, providing the evidence for the validity of this method (by comparing the curves). In this way, we can get the exact β - functions (see also Appendix A). They are in agreement with [3, 4], but not with [10].

5 Conclusions

We find first instanton corrections to the matrix of coupling constants at the symmetric point on the moduli space. Comparing the spectral curves for the theories with different number of flavors, we get some useful relations between couplings and present the proof that the $N_f = 2N_c$ spectral curve could not be presented as a covering of the sphere. Also we propose the strong evidence for the method of [7] (comparing the curves) in the case of the $SU(2)$ gauge group. One can easily extend this technique to the theories with nonzero bare masses, but the requirement for the coupling matrix to be proportional to the classical one imposes some constraints on the masses (or on their symmetric functions $t_k(m) = \sum_{i_1<\cdots<i_k} m_{i_1}\cdots m_{i_k}$). Some β - functions for such theories are collected in the Appendix. It is easy to check that the results turns into (30) if the masses tend to zero.

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Appendix A : Some exact β - functions at nonzero bare masses

- $N_c = N_f = 2$, $m_1 = -m_2 = m$

$$\Lambda^2 = (F^2-9)^{-1}\left[8F^2(u-4m^2)+24u-8\sqrt{8F^4m^2(2m^2-u)+8uF^2(2u-3m^2)}\right]$$

so that

$$\beta = \frac{F^2-9}{FF'}\left[\frac{\left[u-4m^2-\frac{4F^2m^2(2m^2-u)+2u(2u-3m^2)}{\sqrt{2F^4m^2(2m^2-u)+2uF^2(2u-3m^2)}}\right](F^2-9)}{F^2(u-4m^2)+3u-\sqrt{8F^4m^2(2m^2-u)+8uF^2(2u-3m^2)}}-1\right]^{-1}$$

where for the $SU(2)$ group :

$$F = \frac{\theta_2^4(T)+\theta_3^4(T)}{\theta_2^4(T)-\theta_3^4(T)}$$

- $N_c = 2$, $N_f = 3$, $m_1^2+m_2^2 = 2u$, $m_3 = 0$

$$\beta = \frac{F+2}{F'} - \frac{4(F+2)^2\left(u+\frac{\Lambda^2}{64}\right)^2}{27F'(u-m_1^2)^2}$$

$$\frac{F+2}{27}\left(u+\frac{\Lambda^2}{64}\right)^3 = \frac{\Lambda^2}{64}(u-m_1^2)^2$$

where

$$F = \frac{(2\theta_4^4+\theta_2^4)(2\theta_2^4+\theta_4^4)(\theta_4^4-\theta_2^4)}{(\theta_2^8+\theta_4^8+\theta_2^4\theta_4^4)^{\frac{3}{2}}}$$

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